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Giessen, E. van der

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MICROMECHANICAL AND THERMODYNAMIC ASPECTS OF THE PLASTIC SPIN

E. VAN DER GIESSEN

Delft University of Technology

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Abstract—The paper focuses on the concept of plastic spin and its constitutive description in phenomenological theories for macroscopic large strain elastoplasticity. An overview is given of kinematic descriptions of the plastic spin and of constitutive relationships which have been proposed in the literature. Two important classes are distinguished: one in which plastic spin constitutive laws are derived from application of tensor representation theorems, and the second where they are derived from normality rules. Both classes are critically examined from a micromechanical point of view as well as from the standpoint of deterministic irreversible thermodynamics. The role of nonsymmetric state or structure variables in macroscopic plastic spin constitutive relations is emphasized and their origin is identified within the framework of crystal plasticity. The plastic spin in simple shear deformation processes is studied using various kinematic hardening models as well as a simple analytical multiple slip model proposed here.

I. INTRODUCTION

The macroscopic constitutive modeling of large deformation elastoplasticity of polycrystalline materials has attracted a great deal of attention during recent years. Of particular importance is the deformation-induced plastic anisotropy associated with the development of crystallographic textures. Employing an intermediate state type of concept in some form, the plastic spin emerges as an additional concept in the kinematics of elastic-plastic deformations. Although the importance of a constitutive description for the plastic spin was pointed out already by MANDEL [1971] and KRATOCHVIL [1971], large-scale research into appropriate formulations was triggered much later by the observation of NAGTEGAAL and DE JONG [1982] of a strikingly unrealistic response to large simple shear according to a more or less accepted kinematic hardening model. It seems to be generally appreciated now that the plastic spin is of key importance in anisotropic materials and various constitutive laws for the plastic spin have been proposed in recent years, both for kinematic hardening and for other types of persistent or induced anisotropies. However, a number of these approaches seem to be fundamentally different in nature.

For instance, MANDEL [1982] proposed a generalized normality condition for the plastic rate of deformation tensor (the word deformation implying both strain and rotation) with the plastic spin being given by its antisymmetric part. DAFALIAS [1983,1984,1985a, 1985b] and LORET [1983] introduced general constitutive relationships for the plastic spin tensor on the basis of representation theorems for isotropic functions. VAN DER GIESSEN [1990] presented a constitutive theory which employs the additional concept of a plastically induced orientational structure. This theory naturally involves nonsymmetric tensorial internal state variables, which, in turn, govern the plastic spin through a generalized normality condition.

In addition to such purely phenomenological approaches, the macroscopic material behavior of polycrystalline materials has been studied extensively over a number of years using polycrystal models based on crystal plasticity. Excellent overviews of this vast area are given by, for example, HAVNER [1982], ASARO [1983] and ASARO and NEEDLEMAN [1985]. These models potentially give a rigorous description of deformation-induced anisotropy due to texture development. As the constitutive equations for the plastic spin in single crystal plasticity are well-established, such models may provide a physically sound background for the formulation of the macroscopic plastic spin; but systematic studies with this aim seem to be rather scarce as yet.

The purpose of this paper is to collect, compare, and discuss the various phenomenological approaches that may be found in recent literature against the background of the thermodynamics and the micromechanics of large elastoplastic deformations. The thermodynamics of plastic spinning seems to have received very little attention, but it has been emphasized recently by PERZYNA [1988] and VAN DER GIESSEN [1989a,1989b, 1990]. Special attention is given to the implications of the deterministic approaches to irreversible thermodynamics of BIOT [1977,1984] and ZIEGLER [1977] which naturally lead to generalized normality structures. Furthermore, the phenomenological plastic spin constitutive laws are confronted with rigorous physical crystallographic slip theories. The connection of these constitutive descriptions with deterministic thermodynamics is discussed in detail. A relatively simple, analytical multiple slip model for macroscopic deformations is finally presented which provides a convenient means for further micromechanical considerations.

The argument is carried in the direct notation of tensors which are denoted by boldface characters. The tensor product is denoted by $\otimes$ and the following operations apply ($\boldsymbol{a} = a_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, $\boldsymbol{b} = b_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ and $\{\boldsymbol{e}_i\}$ is a set of Cartesian base vectors): $\boldsymbol{ab} = a_{ik}b_{kj}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ and $\boldsymbol{a}:\boldsymbol{b} = a_{ij}b_{ij}$. The scalar product of two tensors of unspecified order is denoted symbolically by $\bullet$. Superscripts T and -1 denote the transpose and inverse of a second-order tensor, respectively, and tr denotes the trace. Subscripts s and a denote the symmetric and antisymmetric parts, respectively, and a superposed dot denotes the material time derivative or rate.

II. KINEMATICS

The concept of an intermediate configuration is now a well-understood tool in finite deformation plasticity theories, but different precise definitions and interpretations may be found in the literature. DAFALLIAS [1987] gives an excellent overview of the kinematic aspects of a number of such concepts and discusses their interrelationships. As far as the kinematics are concerned, two main distinct approaches may be perceived, namely the concept of an isoclinic configuration and that of a rotation-free intermediate configuration.

The concept of an isoclinic configuration was introduced by MANDEL [1971] and used subsequently by, for example, MANDEL [1973,1982], HALPHEN [1975], and LORET [1983]. Motivated by single crystal plasticity, he introduced an orthogonal triad of direct or vectors embedded in the continuum, with the aim of representing the macroscopic substructure of the crystalline material (see also MANDEL [1973]). The orientation of this triad in reference to a spatially fixed global frame of reference defines the orientation of the intermediate unstressed configuration. The isoclinic configuration is then defined as the intermediate configuration in which the triad of director vectors retains the same

orientation in this fixed global frame. More importantly, the introduction of director vectors correctly decouples the motion of the continuum from the kinematics of the materials' substructure (see also MANDEL [1973,1982] and DAFALIAS [1987]). The elastic part $\boldsymbol{F}^e$ of the continuum deformation gradient $\boldsymbol{F}$ includes the elastic stretch as well as the rotation necessary to bring the director triad into the fixed orientation. According to the polar decomposition theorem, $\boldsymbol{F}^e$ can be expressed as $\boldsymbol{F}^e = \boldsymbol{VR} = \boldsymbol{RU}$ with $\boldsymbol{U}$ and $\boldsymbol{V}$ symmetric and $\boldsymbol{R}$ proper orthogonal. With $\boldsymbol{F}^p$ denoting the plastic part of $\boldsymbol{F}$, one can define the plastic rates of deformation and strain and spin as

$$\boldsymbol{A}^p = \dot{\boldsymbol{F}}^p \boldsymbol{F}^{p^{-1}}, \quad \boldsymbol{\Delta}^p = \{\boldsymbol{A}^p\}_s, \quad \boldsymbol{\Omega}^p = \{\boldsymbol{A}^p\}_a, \tag{1}$$

respectively. Based on the decomposition $\boldsymbol{F} = \boldsymbol{F}^e \boldsymbol{F}^p$, the continuum rate of deformation $\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$ and its strain-rate and spin parts $\boldsymbol{d}$ and $\boldsymbol{\omega}$ can be decomposed additively as

$$\boldsymbol{L} = \boldsymbol{L}^e + \boldsymbol{L}^p, \quad \boldsymbol{d} = \boldsymbol{d}^e + \boldsymbol{d}^p, \quad \boldsymbol{\omega} = \boldsymbol{\omega}^e + \boldsymbol{\omega}^p, \tag{2}$$

with the elastic and plastic parts defined by

$$\boldsymbol{L}^e = \dot{\boldsymbol{F}}^e \boldsymbol{F}^{e^{-1}}, \quad \boldsymbol{L}^p = \boldsymbol{F}^e \boldsymbol{A}^p \boldsymbol{F}^{e^{-1}}, \tag{3}$$

$$\boldsymbol{d}^e = \{\boldsymbol{L}^e\}_s, \quad \boldsymbol{\omega}^e = \{\boldsymbol{L}^e\}_a, \quad \boldsymbol{d}^p = \{\boldsymbol{L}^p\}_s, \quad \boldsymbol{\omega}^p = \{\boldsymbol{L}^p\}_a. \tag{4}$$

Within a slightly different framework, the concept of isoclinic configurations was already used earlier by BESSELING [1968]. His approach was followed by VAN DER GIESSEN [1987,1989a]. BESSELING [1968] specified the orientation of the intermediate, so-called natural reference state by means of an orthogonal triad defining the directions of anisotropy, which were assumed to remain invariable. A noteworthy feature of his approach is that the plastic part $\boldsymbol{F}^p$ is not used; instead the plastic rate of deformation $\boldsymbol{A}^p$ is introduced as the primary quantity to characterize the plastic deformation process. The kinematics is then further equivalent to eqns (2) and (3) (see VAN DER GIESSEN [1989a]).

The second class of approaches uses the concept of a rotation-free intermediate configuration (see, e.g., LEE [1969], DAFALIAS [1984,1985a,1985b], AGAH-TEHRANI *et al.* [1987]). This configuration is such that the elastic part $\boldsymbol{V}$ of $\boldsymbol{F}$ is symmetric, $\boldsymbol{V} = \boldsymbol{V}^T$, including only the elastic stretch. Using again MANDEL's [1971] director vectors to specify the orientation of the substructure, the rates of change of material quantities have to be considered relative to this rotating triad (MANDEL [1971,1973]). This implies the use of corotational rates, denoted here by a superposed °, with respect to the spin $\boldsymbol{w}$ of the director vectors. The same kinematic decomposition as in eqn (2) is obtained, but the various rates are now expressed in terms of $\boldsymbol{w}$, $\boldsymbol{V}$ and the plastic part $\boldsymbol{P}$ of $\boldsymbol{F}$ through

$$\boldsymbol{L}^e = \boldsymbol{w} + \mathring{\boldsymbol{V}}\boldsymbol{V}^{-1}, \quad \boldsymbol{L}^p = \boldsymbol{V}\boldsymbol{A}_0^p \boldsymbol{V}^{-1}, \tag{5}$$

instead of eqn (3). Here, the plastic rates of deformation, strain and spin in the intermediate configuration are given by

$$\boldsymbol{A}_0^p = \dot{\boldsymbol{P}}\boldsymbol{P}^{-1}, \quad \boldsymbol{\Delta}_0^p = \{\boldsymbol{A}_0^p\}_s, \quad \boldsymbol{\Omega}_0^p = \{\boldsymbol{A}_0^p\}_a, \tag{6}$$

and $\mathring{\boldsymbol{V}} = \dot{\boldsymbol{V}} - \boldsymbol{w}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{w}$ and $\mathring{\boldsymbol{P}} = \dot{\boldsymbol{P}} - \boldsymbol{w}\boldsymbol{P}$ (see, e.g., DAFALIAS [1987]). Notice that in this formulation the spin $\boldsymbol{w}$ of the director triad enters $\boldsymbol{\omega}^e$ explicitly, whereas it is hidden in the rate of change of $\boldsymbol{F}^e$ when using the isoclinic concept. In the case of small elastic strains, $\boldsymbol{V} \simeq \boldsymbol{I}$ or $\boldsymbol{F}^e \simeq \boldsymbol{R}$, we have either $\boldsymbol{\omega}^e \simeq \boldsymbol{w}$ or $\boldsymbol{\omega}^e \simeq \dot{\boldsymbol{R}}\boldsymbol{R}^T$.

Thus, the spin component $\boldsymbol{\omega}^e$ in (2c) characterizes the spin of the director triad or the substructure, while the plastic spin part $\boldsymbol{\omega}^p$ can be roughly regarded as the spin of the material relative to this substructure. This distinction is well understood in crystal slip processes and will be discussed in more detail later. Actually, the title plastic spin can be conferred upon either the tensor $\boldsymbol{\omega}^p$ in the current configuration or the tensors $\boldsymbol{\Omega}^p$ or $\boldsymbol{\Omega}_0^p$ in either one of the two intermediate configurations considered (see also NEMAT-NASSER [1982]). They are related through the mapping relations (3b) and (5b). The $\boldsymbol{\Omega}^p$ and $\boldsymbol{\Omega}_0^p$ may be considered more closely related to the actual plastic deformation mechanism, but we shall focus here on the plastic spin $\boldsymbol{\omega}^p$ since this measure is of primary importance in the eventual rate constitutive laws. Moreover, $\boldsymbol{\omega}^p$ is independent of the type of intermediate configuration chosen and properly invariant under superimposed rigid body rotations, as discussed in detail by DAFALIAS [1987].

III. OVERVIEW OF CONSTITUTIVE LAWS FOR THE PLASTIC SPIN

Constitutive relations for the plastic spin may be divided into two groups. The first group is based on formulations utilizing tensor functions and the second is based on the application of some generalized normality condition. After a brief exposition of general approaches, plastic spin constitutive relations in kinematic hardening theories are discussed.

Cauchy stress is denoted by $\boldsymbol{\sigma}$, and let $\{\boldsymbol{\alpha}_i\}$ and $\{\boldsymbol{a}_i\}$ be sets of internal state variables which can be tensors, vectors, or scalars. The $\boldsymbol{a}_i$ are defined with respect to the current configuration and the $\boldsymbol{\alpha}_i$ are understood to be properly transported into the intermediate configuration. Attention is confined to time-independent plasticity and temperature effects are omitted.

III.1 *Tensor function formulations*

MANDEL [1971] and KRATOCHVIL [1971] were the first to point out explicitly that constitutive relations are required not only for the plastic strain-rate but also for the plastic spin. The first plastic spin constitutive relation proposed was based on a tensor-valued function $\boldsymbol{B}$ for the plastic rate of deformation in the isoclinic configuration (MANDEL [1971,1973]):

$$\boldsymbol{A}^p = \langle\lambda\rangle \boldsymbol{B}(\boldsymbol{\tau}, \boldsymbol{\alpha}_i) \Rightarrow \boldsymbol{\Omega}^p = \langle\lambda\rangle \{\boldsymbol{B}(\boldsymbol{\tau}, \boldsymbol{\alpha}_i)\}_a, \tag{7}$$

where λ is a properly defined loading index, $\langle\ \rangle$ are the Macauly brackets ($\langle\lambda\rangle = \lambda$ if $\lambda > 0$ and $\langle\lambda\rangle = 0$ if $\lambda \leq 0$) and $\boldsymbol{\tau} = (\det \boldsymbol{F}^e)\boldsymbol{F}^{e^{-1}}\boldsymbol{\sigma}\boldsymbol{F}^{e^{-T}}$ is the Kirchhoff stress in the intermediate configuration. Using eqns (3) and (4c) these expressions can formally be transformed into an expression for the plastic spin tensor $\boldsymbol{\omega}^p$ in the current configuration. DAFALIAS [1985a,1988] showed that $\boldsymbol{\omega}^p$ can in general be expressed by

$$\boldsymbol{\omega}^p = \langle\lambda\rangle \boldsymbol{W}^p(\boldsymbol{\sigma}, \boldsymbol{a}_i), \tag{8}$$

where $\boldsymbol{W}^p$ is an isotropic antisymmetric tensor-valued tensor function of its arguments, provided that the material symmetries (initial, persisting, or induced) are taken into account by a proper choice of some of the $\boldsymbol{a}_i$ and their evolution law. Here it should be noted that the mathematical isotropy of $\boldsymbol{W}^p$ does not imply physical isotropy, due to the tensorial nature of $\boldsymbol{a}_i$ (FARDSHISHEH & ONAT [1974]; DAFALIAS [1985b]).

KRATOCHVIL [1971] seems to have been the first to suggest the application of tensor representation theorems (e.g., WANG [1970]) in order to operationalize the constitutive relations for the plastic spin. Independently, DAFALIAS [1983,1984,1985a,1985b] and LORET [1983] actually used the representation theorems to provide explicit constitutive relations for the plastic strain-rate $\boldsymbol{d}^p$ as well as, but separately, for the plastic spin. In the case of one symmetric second-order tensor $\boldsymbol{a}$ as internal structure variable, both DAFALIAS [1983] and LORET [1983] showed that the plastic spin is in general given by a constitutive relation of the form

$$\begin{aligned}\boldsymbol{\omega}^p = \langle\lambda\rangle\{&\eta_1(\boldsymbol{a}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{a}) + \eta_2(\boldsymbol{a}^2\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{a}^2) + \eta_3(\boldsymbol{a}\boldsymbol{\sigma}^2 - \boldsymbol{\sigma}^2\boldsymbol{a})\\ &+ \eta_4(\boldsymbol{a}\boldsymbol{\sigma}\boldsymbol{a}^2 - \boldsymbol{a}^2\boldsymbol{\sigma}\boldsymbol{a}) + \eta_5(\boldsymbol{\sigma}\boldsymbol{a}\boldsymbol{\sigma}^2 - \boldsymbol{\sigma}^2\boldsymbol{a}\boldsymbol{\sigma})\},\end{aligned} \tag{9}$$

where the η_i's are scalar valued functions of the invariants tr $\boldsymbol{\sigma}$, tr $\boldsymbol{\sigma}^2$, tr $\boldsymbol{\sigma}^3$, tr $\boldsymbol{a}$, tr $\boldsymbol{a}^2$, tr $\boldsymbol{a}^3$, tr$(\boldsymbol{a}\boldsymbol{\sigma})$, tr$(\boldsymbol{a}^2\boldsymbol{\sigma})$, tr$(\boldsymbol{a}\boldsymbol{\sigma}^2)$, tr$(\boldsymbol{a}^2\boldsymbol{\sigma}^2)$ of $\boldsymbol{\sigma}$ and $\boldsymbol{a}$ and any other scalar internal variable (e.g., the accumulated plastic strain). These relations rigorously show that the plastic spin in isotropic material vanishes, as already concluded by MANDEL [1971] and KRATOCHVIL [1971]. Applications to kinematically hardening materials (see further Sec. III.3) as well as to other types of anisotropy such as invariable orthotropy and transverse isotropy were also considered (DAFALIAS [1984,1985a,1985b]; LORET [1983]). Assuming a simple flow rule where $\boldsymbol{d}^p$ is proportional to $\boldsymbol{\sigma}$ or to $\boldsymbol{\sigma} - \boldsymbol{a}$, the lowest order approximation to (9) becomes

$$\boldsymbol{\omega}^p = \tfrac{1}{2}\rho(\boldsymbol{a}\boldsymbol{d}^p - \boldsymbol{d}^p\boldsymbol{a}). \tag{10}$$

Expressions of this type have been used subsequently by several authors, for example, BAMMANN and AIFANTIS [1987], PAULIN and PECHERSKI [1987], and PECHERSKI [1988]. A heuristic derivation of (10) on the basis of a weighted average spin of material unit vectors oriented along eigenvectors of $\boldsymbol{a}$ has been given by DAFALIAS [1985a].

Earlier, HAHN [1974] proceeded along similar lines but adopting a description relative to the intermediate configuration. If the material is plastically isotropic, the internal variables $\boldsymbol{\alpha}_i$ are scalars and, on the basis of representation theorems, HAHN [1974] obtained an expression for the plastic spin of the following form

$$\boldsymbol{\Omega}^p = \langle\lambda\rangle\{\chi_1\boldsymbol{T}_a + \chi_2(\boldsymbol{T}_s\boldsymbol{T}_a + \boldsymbol{T}_a\boldsymbol{T}_s) + \chi_3(\boldsymbol{T}_s\boldsymbol{T}_a^2 - \boldsymbol{T}_a^2\boldsymbol{T}_s)\}, \tag{11}$$

where $\boldsymbol{T}_s$ and $\boldsymbol{T}_a$ are the symmetric and antisymmetric parts of the stress tensor $\boldsymbol{T}$ defined by

$$\boldsymbol{T} = (\det \boldsymbol{F}^e)\boldsymbol{F}^{e^T}\boldsymbol{\sigma}\boldsymbol{F}^{e^{-T}}; \quad \boldsymbol{\sigma} = (\det \boldsymbol{F}^e)^{-1}\boldsymbol{F}^{e^{-T}}\boldsymbol{T}\boldsymbol{F}^{e^T} \tag{12}$$

(see also MANDEL [1971]; VAN DER GIESSEN [1989a]) and where the χ_i's are functions of the invariants tr $\boldsymbol{T}_s$, tr $\boldsymbol{T}_s^2$, tr $\boldsymbol{T}_s^3$, tr $\boldsymbol{T}_a^2$, tr$(\boldsymbol{T}_s\boldsymbol{T}_a^2)$, tr$(\boldsymbol{T}_s^2\boldsymbol{T}_a^2)$, tr$(\boldsymbol{T}_s^2\boldsymbol{T}_a^2\boldsymbol{T}_s\boldsymbol{T}_a)$ of $\boldsymbol{T}$ and

possibly additional scalar variables. In the case of small elastic strains or when the material is also elastically isotropic, the tensor $\boldsymbol{T}$ is symmetric and $\boldsymbol{\Omega}^p$ becomes zero (in the first case, $\boldsymbol{\omega}^p$ vanishes as well).

III.2 *Formulations using generalized normality conditions*

Considering the plastic deformation-rate, $\boldsymbol{A}^p$ in the isoclinic configurations for instance, as the large deformation generalization of the plastic rate of strain in infinitesimal plasticity, HALPHEN [1975] proposed a flow rule in the form of a generalized normality condition (in nine-dimensional space),

$$\boldsymbol{A}^p = \beta_0 \frac{\partial \varphi}{\partial \boldsymbol{T}}; \quad \varphi = \varphi(\boldsymbol{T}, \boldsymbol{\alpha}_i), \tag{13}$$

expressed in terms of the stress tensor $\boldsymbol{T}$ which is dual to $\boldsymbol{A}^p$ (note that $\boldsymbol{T}:\boldsymbol{A}^p$ is the energy dissipation rate per unit volume in the intermediate configuration). Here, the yield function φ is used as a plastic potential and $\beta_0 \geq 0$ if $\varphi = 0$ and $\dot{\varphi} \geq 0$ and $\beta_0 = 0$ if $\varphi < 0$ or $\varphi = 0$ and $\dot{\varphi} < 0$.

MANDEL [1971] had suggested a similar expression within the framework of single crystal plasticity. Later, he used considerations of macroscopic plastic deformations in polycrystals due to crystallographic slip to motivate the use of a normality condition for the rate of deformation tensor $\boldsymbol{L}^p$ in the actual configuration (MANDEL [1982]):

$$\boldsymbol{L}^p = \beta \frac{\partial \varphi}{\partial \boldsymbol{\sigma}} \Rightarrow \boldsymbol{\omega}^p = \beta \left\{ \frac{\partial \varphi}{\partial \boldsymbol{\sigma}} \right\}_a. \tag{14}$$

Restricting to small elastic strains, $\boldsymbol{F}^e \simeq \boldsymbol{R}$, the yield function φ in (14) is considered a function of the stress tensor $\boldsymbol{\tau} \simeq \boldsymbol{R}^T \boldsymbol{\sigma} \boldsymbol{R}$ and $\boldsymbol{\alpha}_i$. Furthermore, φ is supposed to be a desymmetrized expression in $\boldsymbol{\tau}$, while the actual symmetry of $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ is not invoked until after application of (14). MANDEL [1982] demonstrated this approach using a generalized, desymmetrized version of Hill's yield function for a material that is orthotropic in the intermediate configuration. It was shown, using Cartesian coordinates according to the axes of orthotropy, that for instance d_{12}^p and ω_{12}^p are at any instant proportional, a result obtained independently by DAFALIAS [1984] by application of (9).

Continuing along the lines set out by BESSELING [1968,1985], VAN DER GIESSEN [1989a] considered a flow rule similar to eqn (13) and, more particularly, the resulting flow rule (14a) in the current configuration which is in general obtained by invoking eqns (3b) and (12). Here, the yield function φ is taken to be a function of the components of $\boldsymbol{T}$ with respect to the triad of director vectors. This is equivalent to accounting for initial and persistent material symmetries only and using this triad to serve as the internal variables $\boldsymbol{\alpha}_i$. By observing that Cauchy's stress tensor is symmetric from principle, VAN DER GIESSEN [1989a,1990] concluded that $\partial\varphi/\partial\boldsymbol{\sigma}$ is symmetric and, hence, that $\boldsymbol{\omega}^p$ according to (14b) is zero. Clearly this is in contrast with MANDEL [1982] and this will be further discussed in Sec. IV.

As a further elaboration, VAN DER GIESSEN [1989b,1990] proposed a constitutive theory which accounts for deformation-induced anisotropy. In this theory, crystallographic anisotropy or texture is incorporated in a phenomenological manner by endowing the continuum with a so-called plastically induced orientational structure (PIOS). This PIOS

is specified by a (second) triad of directors, $\{\boldsymbol{\pi}_I\}$ in the intermediate configuration, and is taken to be induced by the plastic deformation process. The evolution of the PIOS is proposed to be governed by $\dot{\boldsymbol{\pi}}_I = \boldsymbol{A}^\pi \boldsymbol{\pi}_I$ and the tensor $\boldsymbol{A}^\pi$ is assumed to be related to the plastic rate of deformation $\boldsymbol{A}^p$ according to $\boldsymbol{A}^\pi = \boldsymbol{\Pi} : \boldsymbol{A}^p$ in terms of a fourth-order tensor $\boldsymbol{\Pi}$. The so-called micro-stress tensor $\boldsymbol{M}$ which is conjugate to $\boldsymbol{A}^\pi$ is used as an additional internal variable. VAN DER GIESSEN [1990] continued by assuming a yield function $\varphi(\bar{\boldsymbol{T}}, \boldsymbol{M}, \boldsymbol{\alpha}_i)$, with $\bar{\boldsymbol{T}}$ defined by $\bar{\boldsymbol{T}} = \boldsymbol{T} - \boldsymbol{M} : \boldsymbol{\Pi}$,[1] and pointing out that in general $\boldsymbol{M}$ is an essentially nonsymmetric tensor. Eventually, the flow rule is then given by a normality condition similar to (14a) but operating in $\bar{\boldsymbol{\sigma}}$-space:

$$\boldsymbol{L}^p = \beta \frac{\partial \varphi}{\partial \bar{\boldsymbol{\sigma}}} \Rightarrow \boldsymbol{\omega}^p = \beta \left\{ \frac{\partial \varphi}{\partial \bar{\boldsymbol{\sigma}}} \right\}_a , \tag{15}$$

where [cf. eqn (12b)]

$$\bar{\boldsymbol{\sigma}} = (\det \boldsymbol{F}^e)^{-1} \boldsymbol{F}^e \bar{\boldsymbol{T}} \boldsymbol{F}^{e^T}, \quad \boldsymbol{m} = (\det \boldsymbol{F}^e)^{-1} \boldsymbol{F}^e \boldsymbol{M} \boldsymbol{F}^{e^T}. \tag{16}$$

In contrast with Cauchy stress $\boldsymbol{\sigma}$, the tensor $\bar{\boldsymbol{\sigma}}$ is essentially nonsymmetric, due to the essential nonsymmetry of $\boldsymbol{m}$, and VAN DER GIESSEN [1990] concluded that (15b) provides a nontrivial constitutive relation for the plastic spin. When for a certain material $\boldsymbol{m}$ and $\bar{\boldsymbol{\sigma}}$ are symmetric, the plastic spin vanishes.

III.3 *Application to kinematic hardening*

Most of the plastic spin constitutive relations discussed in the previous two subsections were applied within the context of large deformation kinematic hardening models. DAFALIAS [1983,1985a,1985b] and LORET [1983] used the simplest form of their general proposal (9) for the plastic spin, retaining only the quadratic terms and identifying one of the internal variables $\boldsymbol{a}$ with the symmetric back stress tensor. Using a yield surface specified by $\varphi = \frac{1}{2} \operatorname{tr} \tilde{\boldsymbol{s}}^2 - \frac{1}{3}\sigma_y^2 = 0$, with $\tilde{\boldsymbol{s}} = \boldsymbol{s} - \boldsymbol{a}$ and $\boldsymbol{s}$ the deviatoric part of $\boldsymbol{\sigma}$, and letting the plastic strain-rate be governed by the normality condition, $\boldsymbol{d}^p = \langle \lambda \rangle \sqrt{3/2}\, \tilde{\boldsymbol{s}}/\sigma_y$, their plastic spin constitutive relation can be written as eqn (10) with $\rho = 2\eta\sigma_y\sqrt{2/3}$ (here we basically follow DAFALIAS' [1985a,1985b] notation). Furthermore, they specify the constitutive law for $\boldsymbol{a}$ by, among others, a Ziegler type shift rule,

$$\mathring{\boldsymbol{a}} = \dot{\boldsymbol{a}} - \boldsymbol{\omega}^e \boldsymbol{a} + \boldsymbol{a}\boldsymbol{\omega}^e = \mu \boldsymbol{d}^p \tag{17}$$

in terms of a corotational rate based on the spin $\boldsymbol{\omega}^e = \boldsymbol{\omega} - \boldsymbol{\omega}^p$. Note that when the elastic strains remain small we have $\mathring{\boldsymbol{a}} \simeq \dot{\boldsymbol{a}}$. This rate expresses the assumption that the back stress is embedded in the material substructure, so that the proper corotational rate is with respect to the substructure spin $\boldsymbol{\omega}^e$. It is noted that a similar rate equation for the back stress was proposed by ONAT [1983] without application of the plastic spin concept (see also discussion by DAFALIAS [1988]).

The plastic behavior in this kinematic hardening model is thus determined by the hardening modulus μ and the function ρ in the plastic spin relation (10). DAFALIAS [1985a] and LORET [1983] considered ρ to be constant. On the basis of an analysis of the angu-

[1] In fact, VAN DER GIESSEN [1990] uses a slight variation of $\bar{\mathbf{T}}$, but this is of no importance for the present considerations.

lar velocity in finite simple shear of a material element which may be associated with an induced preferred direction, PAULUN and PECHERSKI [1987] proposed to specify ρ by the function $\rho = \sqrt{3/2}\, 16a_e/(3\mu^2 + 4a_e^2)$ with $a_e = \sqrt{3\mathrm{tr}\, \boldsymbol{a}^2/2}$, thus eliminating ρ as an additional material parameter (see also PECHERSKI [1988]). Various other functions have been proposed by AIFANTIS [1987] for different extensions of pure kinematic hardening models.

VAN DER GIESSEN [1989b,1990] discussed a kinematic hardening model as a special case of the PIOS theory, based on the highly simplified assumption that the PIOS codeforms with the material, that is $\boldsymbol{A}^{\pi} = \boldsymbol{A}^{p}$. The plastic deformation process is then governed by the deviatoric part $\bar{\boldsymbol{s}} = \boldsymbol{s} - \boldsymbol{m}$ of $\bar{\boldsymbol{\sigma}}$ and the yield surface is taken to be specified by $\varphi = \frac{1}{2}\mathrm{tr}\, \bar{\boldsymbol{s}}^2 - \frac{1}{3}\sigma_y^2 = 0$. Based on the assumption that the micro-stress or back stress tensor $\boldsymbol{m}$ is attached to the PIOS and invoking the mixed tensorial nature of $\boldsymbol{m}$, the following evolution law is proposed (VAN DER GIESSEN [1989b])

$$\mathring{\boldsymbol{m}}^{\wedge} = \dot{\boldsymbol{m}} - \boldsymbol{m}\boldsymbol{L}^{T} + \boldsymbol{L}^{T}\boldsymbol{m} + \boldsymbol{m}\,\mathrm{tr}\,\boldsymbol{d} = \mu \boldsymbol{L}^{p^{T}}. \tag{18}$$

It is noted that in this generalization of Ziegler's shift rule, the back stress tensor is essentially nonsymmetric and the constitutive relation for the plastic spin is then given through eqn (15b), that is, $\boldsymbol{\omega}^{p} = \beta \boldsymbol{m}_a$.

In the references cited above, these kinematic hardening models were applied to a material in finite simple shear. Emphasis there was on the predicted stress response, in view of the observation by NAGTEGAAL and DE JONG [1982] of an oscillating shear stress response according to classical finite strain kinematic hardening models (i.e., according to eqn (17) with $\boldsymbol{\omega}^{p} = \boldsymbol{0}$, $\rho = 0$).

In Fig. 1 we briefly study the plastic spin as a function of the applied shear γ along the x^1 direction in the $x^1 - x^2$ plane. In the analysis, small strain (hypo) elasticity is assumed with the rate equation for the Cauchy stress given by $\dot{\boldsymbol{\sigma}} = \mathring{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{L}} : \boldsymbol{d}^{e}$ where $\boldsymbol{\mathcal{L}}$ is

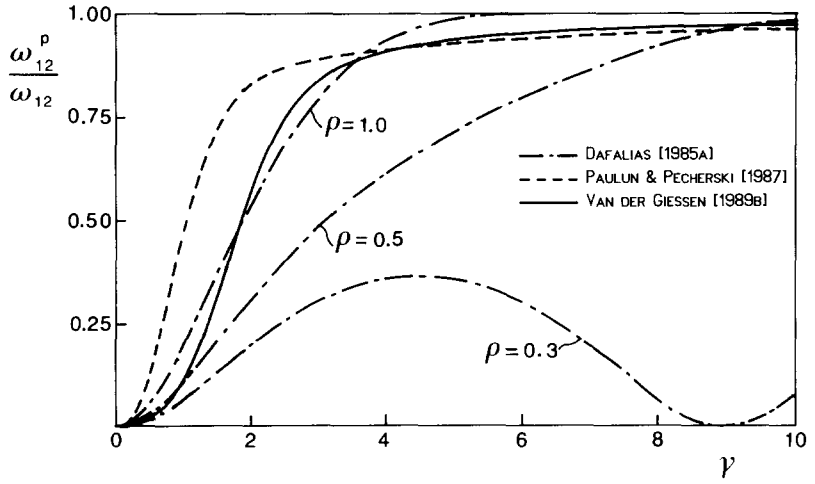

Fig. 1. The plastic spin versus shear strain for various kinematic hardening models. In the theory based on eqns (10) and (17), constant values of ρ (normalized by the yield stress σ_y) as in DAFALIAS [1985a] are considered as well as the function according to PAULUN and PECHERSKI [1987]. The theory of VAN DER GIESSEN [1989b] is based on eqns (15b) and (18).

the tensor of elastic moduli, whose components in the Cartesian basis $\{\boldsymbol{e}_i\}$ associated with the frame x^i are given by $\mathfrak{L}_{ijkl} = C\delta_{ij}\delta_{kl} + G(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il} - \frac{2}{3}\delta_{ij}\delta_{kl})$ with C and G the bulk and shear modulus, respectively. The final rate equations in terms of the continuum motion for the various models can be written on the Jaumann rate form $\overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\omega}\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{\omega} = [\mathfrak{L} - (1-h)\mathfrak{Y}] : \boldsymbol{d}$. Here, the fourth-order tensor $\mathfrak{Y}$ depends on the particular theory; detailed expressions may be found in, for example, DAFALIAS [1985a,1985b] and VAN DER GIESSEN [1989b]. For the case of uniaxial tension, the various constitutive equations correctly simplify to the classical finite strain model, with the parameter h being related immediately to the tangent modulus in the uniaxial true stress-logarithmic strain curve.

Figure 1 shows the nonzero component ω_{12}^p of $\boldsymbol{\omega}^p$ normalized by the continuum spin $\omega_{12} = \dot{\gamma}/2$ versus γ for $G/\sigma_y = 40$ and $h = 0.0123$. With the exception of DAFALIAS' [1985a] model with $\rho = 0.3/\sigma_y$, all curves show a gradual transition of $\omega_{12}^p = 0$ to $\omega_{12}^p \to \omega_{12}$ (or $\omega_{12}^e \to 0$). In particular, the plastic spin predictions according to DAFALIAS [1985a] where $\rho = 1/\sigma_y$, PAULUN and PECHERSKI [1987] and VAN DER GIESSEN [1989b] are qualitatively similar, although the stress responses were found to differ substantially. If the value of ρ is reduced to $\rho = 0.3/\sigma_y$, the plastic spin tends to oscillate, just as the associated shear stress (cf. DAFALIAS [1985a]).

IV. THERMODYNAMIC ASPECTS

Plasticity, as an irreversible thermodynamic process, has to satisfy the laws of thermodynamics. Of importance here is the Second Law or the Clausius–Duhem inequality, which implies the requirement that the energy dissipation-rate is at any instant nonnegative. With $\boldsymbol{\Psi}_i^a$ and $\boldsymbol{\Psi}_i^\alpha$ denoting the thermodynamic forces associated with the internal variables $\boldsymbol{a}_i$ and $\boldsymbol{\alpha}_i$, respectively, standard arguments yield the following expression for the energy dissipation rate $\mathfrak{D}$ per unit volume in the current configuration,[2]

$$\mathfrak{D} = \boldsymbol{\sigma} : \mathbf{d}^p - \boldsymbol{\Psi}_i^a \bullet \dot{\boldsymbol{a}}_i \geq 0, \tag{19}$$

and the expressions

$$\mathfrak{D}_0 = \boldsymbol{T} : \boldsymbol{A}^p - \boldsymbol{\Psi}_i^\alpha \bullet \dot{\boldsymbol{\alpha}}_i = \boldsymbol{T}_0 : \boldsymbol{A}_0^p - \boldsymbol{\Psi}_i^\alpha \bullet \dot{\boldsymbol{\alpha}}_i \geq 0, \tag{20}$$

for the dissipation rate $\mathfrak{D}_0$ per unit volume in the isoclinic and rotation-free intermediate configurations, respectively, where $\boldsymbol{T}_0$ is defined by $\boldsymbol{T}_0 = (\det \boldsymbol{V})\boldsymbol{V}\boldsymbol{\sigma}\boldsymbol{V}^{-1}$ [cf. eqn (12)]. The inequalities in eqns (19) and (20) impose fundamental thermodynamic restrictions on the constitutive relations for the thermodynamic fluxes. Traditionally they would be used as restrictions on the phenomenological coefficients in the constitutive equations for $\boldsymbol{d}^p$ and $\dot{\boldsymbol{a}}_i$ in the case of eqn (19) or $\boldsymbol{A}^p(\boldsymbol{A}_0^p)$ and $\dot{\boldsymbol{\alpha}}_i$ in the case of eqn (20).

Referring to the expression (19) it is noted that, due to the symmetry of Cauchy's stress tensor $\boldsymbol{\sigma}$, only the symmetric part $\boldsymbol{d}^p$ of $\boldsymbol{L}^p$ enters explicitly and not the antisymmetric part $\boldsymbol{\omega}^p$. The identification of the internal variables $\boldsymbol{a}_i$ obviously depends on the particular theory, but examples from the literature use only symmetric tensorial internal state variables (MANDEL [1971,1973]; LORET [1983]; DAFALIAS [1984,1985b]). This means that $\boldsymbol{\omega}^p$ also does not enter (19) implicitly and, hence, that the plastic spin $\boldsymbol{\omega}^p$ does not contribute to the energy dissipation rate.

[2]Attention is restricted to contributions to the energy dissipation rate which are due to plasticity only.

Thus, we observe that, as opposed to the plastic strain-rate $\boldsymbol{d}^p$, the constitutive equation for the plastic spin $\boldsymbol{\omega}^p$, for example, eqn (9), is in these cases exempt from the thermodynamic restrictions imposed by (19). For instance, the kinematic hardening theories of DAFALIAS [1983,1985a] and LORET [1983], discussed in Sec. III, employ the symmetric back stress tensor as the single tensorial state variable and the coefficient ρ in the plastic spin law (10) is indeed not subjected to any thermodynamic restriction. This is indeed an intriguing observation. Apparently, the symmetric and antisymmetric parts of the single kinematic quantity $\boldsymbol{L}^p$ play a completely different role in the thermodynamics of the plastic deformation process. It becomes even more intriguing by recalling the fact that the plastic strain-rate and plastic spin in single slip, as one of the fundamental mechanisms of plasticity, are directly coupled. This will be discussed further in the next section.

In formulations relative to the intermediate configurations, the plastic spin part $\boldsymbol{\Omega}^p$ ($\boldsymbol{\Omega}_0^p$) of $\boldsymbol{A}^p$ ($\boldsymbol{A}_0^p$) does contribute to the energy dissipation rate according to eqn (20) since $\boldsymbol{T}(\boldsymbol{T}_0)$ is nonsymmetric. It should be observed however that this is a sole consequence of the elastic embedding which transports $\boldsymbol{L}^p$ into $\boldsymbol{A}^p(\boldsymbol{A}_0^p)$ and adds an antisymmetric contribution due to the elastic stretching. If the elastic strains remain small we have $\boldsymbol{T} \simeq \boldsymbol{R}^T\boldsymbol{\sigma}\boldsymbol{R} \simeq \boldsymbol{\tau}$ and $\boldsymbol{T}_0 \simeq \boldsymbol{\sigma}$, so that only the symmetric strain-rate parts of $\boldsymbol{A}^p$ and $\boldsymbol{A}_0^p$ contribute to the energy dissipation rate: $\boldsymbol{T}:\boldsymbol{A}^p \simeq \boldsymbol{\tau}:\boldsymbol{\Delta}^p$, $\boldsymbol{T}_0:\boldsymbol{A}_0^p \simeq \boldsymbol{\sigma}:\boldsymbol{\Delta}_0^p$.

Even more sharp conclusions are reached when adopting the strictly deterministic approach to irreversible thermodynamics advocated by BIOT [1977,1984] and ZIEGLER [1977]. In this approach, the energy dissipation rate is attributed a controlling function, rather than a restrictive function. The general outline of the theory is as follows. Let the thermodynamic state be defined by m tensor-valued kinematic variables $\boldsymbol{z}_i$ ($i = 1 \ldots m$). When the energy dissipation rate is given by the scalar product

$$\mathfrak{D} = \boldsymbol{Z}_i \bullet \dot{\boldsymbol{z}}_i, \tag{21}$$

of the thermodynamic fluxes $\dot{\boldsymbol{z}}_i$ and the associated dissipative forces $\boldsymbol{Z}_i$ (where the summation convention is used), it is assumed that all irreversible processes are completely determined by the dissipation function $\Phi'(\boldsymbol{z}_i, \dot{\boldsymbol{z}}_i)$ which is such that at any instant $\mathfrak{D} = \Phi'(\boldsymbol{z}_i, \dot{\boldsymbol{z}}_i)$. Under certain conditions, the dissipation function may be expressed as a function Φ of $\boldsymbol{z}_i$ and $\boldsymbol{Z}_i$, which is more convenient for the present discussion. Furthermore, it is assumed that the dissipation mechanism is such that this dissipation function Φ contains sufficient information to determine the fluxes in a unique manner through the dependence of Φ on the dual dissipative forces. Accordingly, ZIEGLER [1977] postulates the principle of maximal dissipation rate or, equivalently, the orthogonality condition to formulate the rate equations in terms of Φ,

$$\dot{\boldsymbol{z}}_i = \nu \frac{\partial \Phi}{\partial \boldsymbol{Z}_i}, \quad \nu = \left(\boldsymbol{Z}_i \frac{\partial \Phi}{\partial \boldsymbol{Z}_i} \right)^{-1} \Phi. \tag{22}$$

Using a variational Lagrangian formulation, similar results were obtained by BIOT [1977,1984]. It is important here to distinguish between different types of processes, so that the m fluxes $\dot{\boldsymbol{z}}_i$ refer to m possibly different processes. ZIEGLER [1977] then distinguishes between so-called compound processes and complex processes. The latter is the general case, while the entire process is called compound when the individual dissipation functions Φ_i of the elementary processes depend only on the respective forces $\boldsymbol{Z}_i$.

Then the total dissipation function is the sum of the Φ_i's and the orthogonality condition (22) must be understood to hold for each single process.

Thus, in this approach, irreversible thermodynamics ultimately rests upon the notion of duality which, via the energy dissipation, indissolubly links thermodynamic fluxes with thermodynamic forces. ZIEGLER [1977] gives a thorough discussion of the physical motivation and the assumptions involved. Further considerations on the foundation of the approach may be found in his last paper (ZIEGLER & WEHRLI [1987]). A key postulate is the absence of so-called gyroscopic forces; that is, forces which are defined by the condition that they depend on the fluxes in such a way that the associated energy dissipation rate is always zero. Only a few particular examples of gyroscopic forces are known, for example, the Coriolis force in a rotating frame of reference and the velocity dependent part of the Lorentz force in electromagnetic fields. In the absence of these forces, which are also excluded from Onsager's reciprocity relations, the thermodynamic process is called purely dissipative. As a consequence, those variables $\boldsymbol{z}_i$ that are nondissipative, since the associated thermodynamic forces are absent, are not determined by the relations (22). In fact, nondissipative variables are considered to be redundant insofar as irreversible thermodynamics is concerned.

BESSELING [1985] and VAN DER GIESSEN [1989a,1900] adopted this approach in large deformation elastoplasticity for materials that are invariably anisotropic based on a dissipation expression of the form (19) with the $\boldsymbol{a}_i$'s being scalars. In doing so, time-independent plasticity was considered as the limiting case of highly nonlinear creep being governed by a dissipation or creep function. From the fact that the plastic spin $\boldsymbol{\omega}^p$ is nondissipative, as discussed above, it was then concluded from this deterministic approach that $\boldsymbol{\omega}^p$ was left indeterminate by thermodynamics and therefore could be arbitrarily set equal to zero.

Equivalently, this conclusion can also be drawn from the flow rule (14a) for $\boldsymbol{L}^p$ by observing that $\{\partial\varphi/\partial\boldsymbol{\sigma}\}_a = 0$ due to the symmetry of $\boldsymbol{\sigma}$. It is interesting to note here that MANDEL [1982] follows similar lines but in his approach, motivated by single crystal considerations, the symmetry of $\boldsymbol{\sigma}$ is temporarily renounced. When the symmetry properties of $\boldsymbol{\sigma}$ are invoked after application of the flow rule (14), $\{\partial\varphi/\partial\boldsymbol{\sigma}\}_a$ and $\boldsymbol{\omega}^p$ do not vanish indeed. This is easily demonstrated by considering the yield function $\varphi = \mathrm{tr}[\boldsymbol{\sigma}(\boldsymbol{n} \otimes \boldsymbol{v})] - \tau_0$, where $\boldsymbol{v}$ and $\boldsymbol{n}$ are vectors, which yields $\boldsymbol{\omega}^p = \beta\{\boldsymbol{v} \otimes \boldsymbol{n}\}_a$. Adopting the deterministic approach to thermodynamics instead, the yield function would read $\varphi' = \mathrm{tr}[\boldsymbol{\sigma}\{\boldsymbol{n} \otimes \boldsymbol{v}\}_s] - \tau_0$, the values of φ and φ' are at any instant equal, but their derivatives with respect to $\boldsymbol{\sigma}$ are not, so that $\boldsymbol{\omega}^p = \boldsymbol{0}$ in the latter case. In conclusion, when the symmetry of $\boldsymbol{\sigma}$ is recognized as a fundamental property which is to be taken into account at all times, MANDEL's [1982] approach conflicts with thermodynamic determinism; but, using the above yield function φ, it gives a correct description of the plastic spin in single crystal plasticity, as will be discussed further in the next section.

It seems worthwhile to point out again that thermodynamic determinism renders the plastic spin $\boldsymbol{\omega}^p$ a redundant quantity when all tensorial structure variables are symmetric. This is a consequence of the assumption that plasticity is a purely dissipative process (ZIEGLER [1977]). This assumption is made plausible by the physics of the process (see Sec. V); whether the assumption holds on a macroscopic scale will have to be verified by numerical or laboratory experiments.

The argument centers around the symmetry of the state variables. Inverting the argument, the deterministic approach seems to suggest that if, from a physical point of view, the plastic spin for the material under consideration is not redundant, this has to

be taken into account by introducing appropriate nonsymmetric structure variables into the theory. This philosophy is followed by VAN DER GIESSEN [1989b,1990] in the theory employing the PIOS concept. Since the PIOS directors $\boldsymbol{\pi}_I$ are introduced to specify both the orientation and intensity of the induced anisotropy, the evolution tensor $\boldsymbol{A}^{\pi}$ contains an essential antisymmetric part which describes the spin of the PIOS. The microstress tensor $\boldsymbol{M}$ is introduced as the dissipative force associated with $\boldsymbol{A}^{\pi}$, so that $\mathfrak{D}_0 = \boldsymbol{T}:\boldsymbol{A}^{p} - \boldsymbol{M}:\boldsymbol{A}^{\pi} - \boldsymbol{\Psi}_i^{\alpha} \bullet \dot{\boldsymbol{\alpha}}_i$, and therefore constitutes in general an essentially nonsymmetric internal state variable. With the assumption $\boldsymbol{A}^{\pi} = \boldsymbol{\Pi}:\boldsymbol{A}^{p}$, the stress tensor $\bar{\boldsymbol{T}} = \boldsymbol{T} - \boldsymbol{M}:\boldsymbol{\Pi}$ becomes the dissipative force associated with $\boldsymbol{A}^{p}$ and the antisymmetric part of $\bar{\boldsymbol{\sigma}}$ (see eqn (16)) constitutes the thermodynamic force conjugate to the plastic spin $\boldsymbol{\omega}^{p}$. Conversely $\boldsymbol{\omega}^{p}$ is governed by $\bar{\boldsymbol{\sigma}}_a$ by virtue of the constitutive relation (15) which is in accordance with thermodynamic determinism.

Also PERZYNA [1988] suggests the application of nonsymmetric state variables. Motivated by crystalline plasticity (cf. also Sec. V), he proposes to use the antisymmetric plastic rotation tensor $\boldsymbol{\theta}^{p}$, defined by $\boldsymbol{\theta}^{p} = \int \boldsymbol{F}^{T}\boldsymbol{\omega}^{p}\boldsymbol{F}\,dt$, as an internal state variable along with the plastic strain measure $\boldsymbol{\epsilon}^{p} = \int \boldsymbol{F}^{T}\boldsymbol{d}^{p}\boldsymbol{F}\,dt$ and two hardening variables $\boldsymbol{\alpha}$ and κ. Assuming that the free energy ψ per unit mass is a state function of these variables, it follows that the plastic spin adds a contribution $-\dot{\boldsymbol{\theta}}^{p}:(\partial\psi/\partial\boldsymbol{\theta}^{p})$ to the energy dissipation rate. The tensor $\boldsymbol{T}_M = \partial\psi/\partial\boldsymbol{\theta}^{p}$ is antisymmetric and has been interpreted by PERZYNA [1988] as a couple stress tensor. Furthermore, he concludes that if $\boldsymbol{T}_M = \boldsymbol{0}$, the plastic rotation cannot play the role of an internal state variable; this is in accord with our strict deterministic standpoint to thermodynamics.

V. MICROMECHANICAL ASPECTS

It should be realized that the issue of the present discussion is the macroscopic plastic spin experienced by a macroscopic volume element. As discussed before, this plastic spin characterizes the spin of this element relative to the substructure. Roughly speaking, the substructure of a polycrystalline material is related to some average grain orientation. The physical origin of the macroscopic plastic spin in a polycrystalline material is to be attributed mainly to the complex and interacting processes of slip in the constituent crystals and, at a larger scale, the gradual rotation of the crystallites themselves. These processes embody the fundamental mechanisms for inelastic deformations and texture development in polycrystalline material. This is the subject of crystal and polycrystal plasticity, a field which has been explored extensively over many years. Recent accounts may be found in HAVNER [1982], ASARO [1983], ASARO and NEEDLEMAN [1985], and AIFANTIS [1987].

V.1 *Single slip*

Single slip is one of the best understood mechanisms for inelastic deformations of elastic–plastic crystals. For finite deformations, the kinematics has been interpreted in terms of the multiplicative decomposition of the deformation gradient tensor (cf. Sec. II) by MANDEL [1971] and RICE [1971]. Accordingly, the deformation gradient is decomposed as $\boldsymbol{F} = \boldsymbol{F}^{e}\boldsymbol{F}^{p}$, where $\boldsymbol{F}^{p}$ describes the plastic shear flow of the material on the current slip system and where $\boldsymbol{F}^{e}$ characterizes the elastic distortion of the crystal lattice with its embedded material along with the rigid rotation (see Fig. 29 in ASARO [1983]). Note that this interpretation follows the isoclinic concept.

For single slip, the plastic shear flow determines the plastic rate of deformation as

$$\boldsymbol{A}^p = \dot{\boldsymbol{F}}^p \boldsymbol{F}^{p^{-1}} = \dot{\gamma} \boldsymbol{v}_0 \otimes \boldsymbol{n}_0, \tag{23}$$

where $\dot{\gamma}$ is the shear rate and where $\boldsymbol{v}_0$ and $\boldsymbol{n}_0$ are orthonormal vectors in the slip direction and normal to the slip plane, respectively, all being measured in the reference configuration. As proposed first by RICE [1971], the vectors $\boldsymbol{v}_0$ and $\boldsymbol{n}_0$ are embedded elastically, so that they are transported into $\boldsymbol{v} = \boldsymbol{F}^e \boldsymbol{v}_0$ and $\boldsymbol{n} = \boldsymbol{n}_0 \boldsymbol{F}^{e^{-1}}$ in the current configuration, respectively. Hence,

$$\boldsymbol{L}^p = \boldsymbol{F}^e \boldsymbol{A}^p \boldsymbol{F}^{e^{-1}} = \dot{\gamma}\, \boldsymbol{v} \otimes \boldsymbol{n} \Rightarrow \boldsymbol{\omega}^p = \dot{\gamma} \{\boldsymbol{v} \otimes \boldsymbol{n}\}_a \tag{24}$$

and the plastic spin is immediately given in terms of the slip plane geometry and the shear rate. DAFALIAS [1988] discussed the constitutive relations for single slip within a macroscopic framework based on a rotation-free intermediate configuration.

It is interesting to note that the normal vector $\boldsymbol{n}$ in the expression (24b) for $\boldsymbol{\omega}^p$ may be eliminated by means of the relation for $\boldsymbol{d}^p$ from (24a) to give

$$\boldsymbol{\omega}^p = (\boldsymbol{v} \otimes \boldsymbol{v})\boldsymbol{d}^p - \boldsymbol{d}^p(\boldsymbol{v} \otimes \boldsymbol{v}). \tag{25}$$

Here we have assumed small elastic distortions so that $\boldsymbol{F}^e$ reduces to an orthogonal tensor $\boldsymbol{R}$ describing the rigid rotation of the crystal lattice. Alternatively, one may eliminate $\boldsymbol{v}$ and express $\boldsymbol{\omega}^p$ in terms of $\boldsymbol{n}$ and $\boldsymbol{d}^p$. Identifying the tensor $\boldsymbol{a} = \boldsymbol{v} \otimes \boldsymbol{v}$ as a structure variable in an internal variable theory, the plastic spin relation (25) is seen to be similar in form to the phenomenological relationship in eqn (10) proposed by DAFALIAS [1983,1985a,1985b] and LORET [1983] for kinematic hardening.

Presenting earlier work as published in DAFALIAS and AIFANTIS [1990], AIFANTIS [1987] then proceeded even further by eliminating $\boldsymbol{v} \otimes \boldsymbol{v}$ from eqn (25) by means of the constitutive equation $\boldsymbol{T}^D = t_m \{\boldsymbol{v} \otimes \boldsymbol{n}\}_s + t_n (\boldsymbol{n} \otimes \boldsymbol{n})$ for the (symmetric) so-called dislocation stress tensor $\boldsymbol{T}^D$ (t_m and t_n are scalar valued material functions). This constitutive relation was proposed at a micro scale on the basis of dislocation theories. The resulting expression for the plastic spin is

$$\boldsymbol{\omega}^p = -t_n^{-1}(\boldsymbol{T}^D \boldsymbol{d}^p - \boldsymbol{d}^p \boldsymbol{T}^D), \tag{26}$$

which is fully similar to the lowest order general relationship according to eqns (9) and (10) as proposed by DAFALIAS [1983] and LORET [1983]. Moreover, by using a scale invariance argument, AIFANTIS [1987] argues that this expression also holds on a macroscopic scale when $\boldsymbol{T}^D$ is then identified as the macroscopic back stress.

MANDEL [1971,1982] discussed the constitutive description of single slip plasticity in terms of a generalized normality condition. With τ_0 denoting the resolved shear stress yield threshold, the well-known Schmid law is expressed via the yield function $\varphi = \mathrm{tr}[\boldsymbol{T}(\boldsymbol{n}_0 \otimes \boldsymbol{v}_0)] - \tau_0$. The plastic rate of deformation $\boldsymbol{A}^p$, eqn (23), is then recovered by application of the generalized normality condition (13) and identifying β_0 with $\dot{\gamma}$ (MANDEL [1971]). Alternatively, the yield function may be written as

$$\varphi = (\det \boldsymbol{F}^e)\mathrm{tr}[\boldsymbol{\sigma}(\boldsymbol{n} \otimes \boldsymbol{v})] - \tau_0, \tag{27}$$

in terms of quantities related to the current configuration and $\boldsymbol{L}^p$, eqn (24), is recovered by application of (14) (MANDEL [1982]). Thus, the plastic strain-rate and plastic spin are correctly described by means of a normality structure along with a yield function of the form (27); but it is essential that the symmetry properties of the Cauchy stress $\boldsymbol{\sigma}$ are omitted. If one would indeed invoke these symmetry properties in the formulation of φ, as discussed in Sec. IV, one would find $\boldsymbol{d}^p \propto \{\boldsymbol{\nu} \otimes \boldsymbol{n}\}_s$ and $\boldsymbol{\omega}^p = \boldsymbol{0}$ from eqn (14). Recognizing this difficulty, MANDEL [1982] argued that since the slip rate $\dot{\gamma}$ depends only on $\boldsymbol{\sigma}$ through the resolved shear stress $\tau = \mathrm{tr}[\boldsymbol{\sigma}\{\boldsymbol{n} \otimes \boldsymbol{\nu}\}_s]$, one could imagine adding virtual nonsymmetric variations to $\boldsymbol{\sigma}$. Note that the nonsymmetry of the stress tensor $\boldsymbol{T}$ in the formulation relative to the unstressed configuration is only a subsidiary effect of the elastic embedding (see also MANDEL [1971,1982]).

It is worthwhile at this point to link these considerations to the deterministic thermodynamic framework discussed in Sec. IV. Assuming small elastic distortions for simplicity, the energy dissipation rate due to the slip process is $\mathfrak{D} = \tau\dot{\gamma}$. Using eqn (24a), this can be expressed in terms of the kinematic variable $\boldsymbol{L}^p$ as

$$\mathfrak{D} = (\tau\boldsymbol{\nu} \otimes \boldsymbol{n}) : \boldsymbol{L}^p = \{\tau\boldsymbol{\nu} \otimes \boldsymbol{n}\}_s : \boldsymbol{d}^p + \{\tau\boldsymbol{\nu} \otimes \boldsymbol{n}\}_a : \boldsymbol{\omega}^p, \tag{28}$$

which shows that the work conjugate to $\boldsymbol{L}^p$ is in this case an essentially unsymmetric tensor. In this formulation, $\boldsymbol{L}^p$ should be considered an independent variable, while the constitutive characteristics are embodied in the tensor $\tau\boldsymbol{\nu} \otimes \boldsymbol{n}$. After decomposition in symmetric and antisymmetric parts, it is observed that the plastic spin adds a nonvanishing contribution to the energy dissipation. We will proceed by showing how the expression (28) can be cast into the general form (19) according to the internal variable theory. First we note that for single slip we have the identity $\boldsymbol{\sigma} : \boldsymbol{d}^p = 2\{\tau\boldsymbol{\nu} \otimes \boldsymbol{n}\}_s : \boldsymbol{d}^p$ by virtue of eqn (24a). Using this to eliminate $\{\tau\boldsymbol{\nu} \otimes \boldsymbol{n}\}_s$ from eqn (28), we find that the dissipation rate for a given rate of deformation $\boldsymbol{L}^p$ and a given stress state $\boldsymbol{\sigma}$ is given by

$$\mathfrak{D} = \boldsymbol{\sigma} : \boldsymbol{d}^p - \boldsymbol{c} : \boldsymbol{L}^p = \bar{\boldsymbol{\sigma}} : \boldsymbol{L}^p; \quad \bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{c}, \tag{29}$$

where $\boldsymbol{c}$ is the unsymmetric tensor defined by $\boldsymbol{c} = \tau(\boldsymbol{n} \otimes \boldsymbol{\nu})$. The second member of (29) is of the general form (19) when $\dot{\boldsymbol{a}}_i$ and $\boldsymbol{\Psi}_i^a$ are identified with $\boldsymbol{L}^p$ and $\boldsymbol{c}$, respectively. In this formulation, the constitutive characteristics of single slip are taken into account by means of the internal structure variable $\boldsymbol{c}$ which naturally emerges as a nonsymmetric tensor. The symmetry conditions on $\boldsymbol{\sigma}$ are implicitly fulfilled and the dissipation tensor dual to $\boldsymbol{L}^p$ is $\bar{\boldsymbol{\sigma}}$. Adopting the deterministic approach to irreversible thermodynamics, eqn (29) moreover suggests the yield function according to

$$\varphi = (\det \boldsymbol{F}^e)\mathrm{tr}(\bar{\boldsymbol{\sigma}}(\boldsymbol{n} \otimes \boldsymbol{\nu})] - \tau_0, \tag{30}$$

instead of formula (27), where, in a given state $\bar{\boldsymbol{\sigma}}$, a flow rule according to eqn (14a) yields $\boldsymbol{L}^p$.

In fact, VAN DER GIESSEN [1990] has followed these lines in the macroscopic constitutive formulation for a rigid-plastic layered material within the framework of the PIOS theory. Basically, the model is a continuum slip model; the material is taken to deform mainly by slip along the layers, but additional plastic normal strain-rates are also possible. The behavior of such material subjected to simple shear was analyzed by DAFALIAS [1984], who also suggested the visualization as a deck of cards.

The current configuration, where the layers are oriented at an angle α with respect

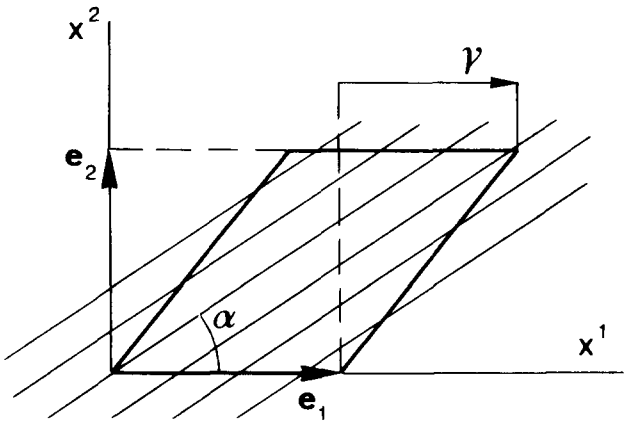

Fig. 2. Simple shear of a layered material by slip along parallel planes tilted at an angle α.

to the x^1 axis of a global Cartesian frame, is shown in Fig. 2. For this material the PIOS can be simply characterized by a spatially fixed orthonormal triad of directors $\boldsymbol{\pi}_I$, such that $\boldsymbol{A}^\pi \equiv \boldsymbol{0}$. Using isoclinic intermediate configurations, the orientation of $\{\boldsymbol{\pi}_I\}$ is arbitrary; for convenience, they are taken to coincide with the global basis $\{\boldsymbol{e}_i\}$ associated with the x^i frame, so that $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are tangential and normal to the slip planes, respectively. The rotation over α which is necessary to bring $\{\boldsymbol{\pi}_I\}$ in accordance with the current tangential and normal directions of the slip planes is determined by $\boldsymbol{F}^e \simeq \boldsymbol{R}$.

The fact that shear deformation can occur only by slip along the slip planes is taken into account by imposing the constraint $A^p_{21} \equiv 0$ on the components of $\boldsymbol{A}^p$ in $\{\boldsymbol{\pi}_I\}$, that is, $\boldsymbol{A}^p = A^p_{IJ}\boldsymbol{\pi}_I \otimes \boldsymbol{\pi}_J$. This kinematic constraint along with the condition $\boldsymbol{A}^\pi \equiv \boldsymbol{0}$ can be taken into account in the PIOS theory by making the constitutive assumption $\Pi_{2121} = 1$ and $\boldsymbol{\Pi}_{IJKL} = 0$ otherwise for the components of $\boldsymbol{\Pi}$, $\boldsymbol{\Pi} = \Pi_{IJKL}\boldsymbol{\pi}_I \otimes \boldsymbol{\pi}_J \otimes \boldsymbol{\pi}_K \otimes \boldsymbol{\pi}_L$. The components of the dissipation tensor $\bar{\boldsymbol{T}} = \boldsymbol{T} - \boldsymbol{M}:\boldsymbol{\Pi} = \bar{T}_{IJ}\boldsymbol{\pi}_I \otimes \boldsymbol{\pi}_J$ are then given by $\bar{T}_{21} = T_{21} - M_{21}$, $\bar{T}_{IJ} = T_{IJ}$ otherwise, which clearly shows the nonsymmetry of $\bar{\boldsymbol{T}}$. The micro-stress tensor M_{21} has been interpreted as the multiplier for the constraint $A^p_{21} \equiv 0$ and is at any instant equal to the applied shear stress $T_{12} = T_{21}$. From a physical point of view, the M_{21} component may be viewed upon as representing, in the usual continuum sense, the normal stress distribution at the interfaces between adjacent layers which enforces the slip motion and prevents "tilting" of the layers (VAN DER GIESSEN [1987]).

The connection with the crystallographic single slip considerations above is immediately clear after identifying $(\boldsymbol{\pi}_1, \boldsymbol{\pi}_2)$ with $(\boldsymbol{\nu}_0, \boldsymbol{n}_0)$ and $(\boldsymbol{R\pi}_1, \boldsymbol{R\pi}_2)$ with $(\boldsymbol{\nu}, \boldsymbol{n})$. Then the dissipation tensor in the current configuration is $\bar{\boldsymbol{\sigma}} = \boldsymbol{R}\bar{\boldsymbol{T}}\boldsymbol{R}^T = \boldsymbol{\sigma} - \boldsymbol{m}$ [cf. eqn (16)] with $\boldsymbol{m} = \boldsymbol{RMR}^T = M_{21}(\boldsymbol{R\pi}_2) \otimes (\boldsymbol{R\pi}_1)$, which is equivalent to the tensor $\bar{\boldsymbol{\sigma}}$ in eqn (29).

When such a layered material is subjected to simple shear at a rate $\dot{\gamma}$ along the x^1 axis, the kinematic constraint $A^p_{21} \equiv 0$ yields the evolution relation $\dot{\alpha} = \frac{1}{2}\dot{\gamma}(\cos 2\alpha - 1)$ for the angle α, that is, for the rotation rate of the substructure. For future reference we note that the solution of this differential equation reads

$$\alpha = \arctan(\gamma + \cot \alpha_0)^{-1}, \tag{31}$$

if α_0 is the initial orientation of the layers (VAN DER GIESSEN [1989c]). The same evolution relation was obtained earlier by DAFALIAS [1984] in a formulation employing ro-

tation-free intermediate configurations. It is of importance to realize that for arbitrary α, normal plastic strain-rates in $\boldsymbol{v}$ and $\boldsymbol{n}$ directions are necessary to accommodate the applied simple shear.

In the foregoing analysis, a nonsymmetric (internal) back stress type variable $\boldsymbol{c}$ emerges as a tool to enforce the essential kinematic constraints involved in single slip [cf. eqn (29)]. A nonsymmetric back stress in the single slip configuration that is of a completely different nature and origin was considered by DAFALIAS and AIFANTIS [1990]. Within the framework of dislocation theory they discuss the case where the dislocation stress tensor also has an antisymmetric part which is given by $\boldsymbol{T}_a^D = t_\omega\{\boldsymbol{v} \otimes \boldsymbol{n}\}_a$ with t_ω a material parameter depending on the dislocation density and the mean free path of dislocations. Assuming then a plausible evolution relation for t_ω, it is found that t_ω and, therefore, $\boldsymbol{T}_a^D$ decay exponentially with the shear strain. They interpret this result as a justification of the use of symmetric back stresses. In view of the considerations in the present paper, the tensor $\boldsymbol{T}_a^D$ may be regarded as a (decaying) correction on the tensor $\boldsymbol{c}$ defined in eqn (29b) (note that if $t_\omega = -\tau$ we have $\boldsymbol{c}_a = \boldsymbol{T}_a^D$).

V.2. *Single crystal and polycrystal deformations*

Individual crystals are generally envisaged to deform by shear along the available slip systems of the crystal. For instance, face-centered cubic (FCC) crystals possess 12 slip systems. The total plastic rate of deformation in a single crystal is then simply given by the sum of the plastic rates of deformation associated with single slip on each slip system α as discussed above. With reference to eqn (24), we find

$$\boldsymbol{L}^p = \sum_\alpha \dot{\gamma}^{(\alpha)} \boldsymbol{v}^{(\alpha)} \otimes \boldsymbol{n}^{(\alpha)} \Rightarrow \boldsymbol{\omega}^p = \sum_\alpha \dot{\gamma}^{(\alpha)} \{\boldsymbol{v}^{(\alpha)} \otimes \boldsymbol{n}^{(\alpha)}\}_a, \tag{32}$$

where an index α indicates a quantity in slip system α (e.g., RICE [1971]; ASARO [1983]). Here the sum runs over the n active slip systems. Assuming rate insensitive slip, the choice of the active slip systems may become nonunique, depending on the slip-hardening law; rate dependent formulations however are inherently free from such ambiguity problems (see, e.g., ASARO [1983]). We have focused on time-independent plasticity here, so we will assume that the active slip systems are known.

It has been shown before in eqn (25) that in the case of single slip the plastic spin can be related linearly to the plastic strain-rate. Evidently this can be done as well for each of the contributions of the individual slip systems. Now, let $\boldsymbol{d}_{(\alpha)}^p$ be the contribution of slip system α to $\boldsymbol{d}^p$, so that $\boldsymbol{d}^p = \sum_\alpha \boldsymbol{d}_{(\alpha)}^p$, and let $\Delta\boldsymbol{d}_{(\alpha)}^p$ be the deviation from the average contribution of slip system α: $\Delta\boldsymbol{d}_{(\alpha)}^p = \boldsymbol{d}_{(\alpha)}^p - \boldsymbol{d}^p/n$. Furthermore, assume that the difference $\Delta\boldsymbol{d}_{(\alpha)}^p$ can be related to the average value $\boldsymbol{d}^p$ by means of the linear mapping relation $\Delta\boldsymbol{d}_{(\alpha)}^p = \boldsymbol{D}^{(\alpha)} : \boldsymbol{d}^p$ in terms of a fourth-order tensor $\boldsymbol{D}^{(\alpha)}$, which will in general depend on all current slip rates and slip plane characteristics. The slip plane normals in eqn (32b) can then be eliminated to give

$$\begin{aligned}\boldsymbol{\omega}^p = &\sum_\alpha \frac{1}{n} (\boldsymbol{v}^{(\alpha)} \otimes \boldsymbol{v}^{(\alpha)})\boldsymbol{d}^p - \boldsymbol{d}^p \sum_\alpha \frac{1}{n} (\boldsymbol{v}^{(\alpha)} \otimes \boldsymbol{v}^{(\alpha)}) \\ &+ \sum_\alpha (\boldsymbol{v}^{(\alpha)} \otimes \boldsymbol{v}^{(\alpha)})(\boldsymbol{D}^{(\alpha)} : \boldsymbol{d}^p) - \sum_\alpha (\boldsymbol{D}^{(\alpha)} : \boldsymbol{d}^p)(\boldsymbol{v}^{(\alpha)} \otimes \boldsymbol{v}^{(\alpha)}).\end{aligned} \tag{33}$$

It is observed that with the identification $\boldsymbol{a} \propto \sum_\alpha (\boldsymbol{\nu}^{(\alpha)} \otimes \boldsymbol{\nu}^{(\alpha)})$, the first two terms in the rhs are similar to the phenomenological relationship (10). But the remaining terms constitute a complicated additional contribution. The practical importance of these terms is as yet unclear, but it is not expected to be negligible.

It is tempting to compare this result with the general macroscopic relation (9) in terms of the symmetric structure variable $\boldsymbol{a}$. Assuming that for the overall crystal deformations we can specify $\boldsymbol{d}^p \propto \boldsymbol{\sigma}$ with reasonable accuracy, as in J_2-like macroscopic theories, eqn (33) is linearly dependent on $\boldsymbol{\sigma}$, that is, $\eta_3 = \eta_5 = 0$ in eqn (9). In general it will not be possible with the above identification of $\boldsymbol{a}$ to express the additional third and fourth terms in (33) in the form of the generators associated with η_2 and η_4 in eqn (9). Hence, it must be expected that a single symmetric structure variable will not suffice to give an adequate description of the single crystal plastic spin.

Supposing again that the active slip systems are known, the constitutive description of rate independent single slip using a normality structure, as discussed in the previous subsection, can be formally extended to multiple slip by employing a finite strain generalization of KOITER's [1953] multiple yield surface theory. This was discussed in detail by MANDEL [1971]. Taking for each slip system a yield function of the form (27), $\varphi^{(\alpha)}(\boldsymbol{\sigma}, \boldsymbol{\nu}^{(\alpha)}, \boldsymbol{n}^{(\alpha)})$, the total plastic strain-rate for the crystallite is obtained from $\boldsymbol{L}^p = \sum_\alpha \beta^{(\alpha)} \partial \varphi^{(\alpha)} / \partial \boldsymbol{\sigma}$ as a generalization of eqn (14). Such an approach needs to be generalized slightly further when one invokes the yield formulation in the form (30) according to deterministic thermodynamics. Then $\varphi^{(\alpha)}$ must be considered a function of $\bar{\boldsymbol{\sigma}}^{(\alpha)} = \boldsymbol{\sigma} - \boldsymbol{c}^{(\alpha)}$ and the flow rule must be specified as $\boldsymbol{L}^p = \sum_\alpha \beta^{(\alpha)} \partial \varphi^{(\alpha)} / \partial \bar{\boldsymbol{\sigma}}^{(\alpha)}$. Multiple slip in a single crystal is hence considered as a compound process (ZIEGLER [1977]). Thus, also along this route a rigorous derivation of the plastic spin relation for single crystals can be obtained, but again it is unclear how the results can be used to construct practically applicable macroscopic constitutive laws.

The extension of single crystal plasticity theories to the description of the behavior of polycrystalline aggregates is a natural though formidable challenge. The key difficulty is how to give an adequate accounting of the grain interactions due to the heterogeneity of the deformation pattern at a microscale. The two classical and extreme assumptions are the Taylor assumption where all grains in the aggregate are subjected to the same homogeneous deformation field and the Sachs principle where all grains experience the same state of stress (see, e.g., HAVNER [1982] and ASARO & NEEDLEMAN [1985]). More recently, "relaxed constraint" models (CANOVA, KOCKS, & JONAS [1984]) and various self-consistent theories (e.g., BERVEILLER & ZAOUI [1979]) have been proposed to improve the way in which grain interactions are taken into account.

These polycrystal models are inherently capable of describing the development of textures. The rate of rotation of the crystal lattice of each grain is determined by the velocity gradient and the plastic spin $\boldsymbol{\omega}^p_{(g)}$ according to eqn (32), and the orientation of the grains during the process can be traced by integration of this lattice spin. In fact, polycrystal models, in particular the Taylor model, have been widely used in the past to study texture development in metals. Broadly speaking, the predicted textures are generally in qualitative agreement with experimental observations but are mostly too sharp (e.g., CANOVA *et al.* [1984]; TOKUDA & YAMADA [1988]; HARREN, LOWE, ASARO, & NEEDLEMAN [1989]). A first comparison of Taylor models with full solutions over a polycrystalline domain has been given very recently by HARREN and ASARO [1989].

From the foregoing it follows that polycrystal models potentially provide a physically well-founded tool for studying plastic spin effects in macroscopic deformations of poly-

crystalline solids. However, applications have so far primarily focused on the development of textures and the macroscopic stress-strain behavior. Specific studies of the macroscopic plastic spin are scarce in the literature. One of the most notable contributions is due to MANDEL [1982]. In this work he used HILL's [1972] averaging procedures to rigorously derive the spin of his macroscopic director triad for the polycrystal, that is, $\boldsymbol{\omega}^e \simeq \boldsymbol{\omega}$ (assuming small elastic strains), from the lattice rotations of the constituent grains. From equilibrium and continuity considerations at a microscale he showed that the only admissible simple possibility is to evaluate $\boldsymbol{\omega}^e \sim \boldsymbol{w}$ as the volume average of the crystals' lattice spins. The corresponding evaluation of the overall plastic spin reads

$$\boldsymbol{\omega}^p = \frac{1}{V}\int_V \boldsymbol{\omega}^p_{(g)}\, dV, \tag{34}$$

where V is the volume of the aggregate and $\boldsymbol{\omega}^p_{(g)}$ is the plastic spin of the grains according to eqn (32b). This calculation can be carried out in principle by means of polycrystal models. Even then, however, the derivation of macroscopic constitutive laws for the plastic spin remains an unaccomplished, formidable task. Similar conclusions are reached by TOKUDA and YAMADA [1988].

V.3. *An analytical multiple slip model*

In this subsection we will briefly discuss the implications of the multiple slip model recently proposed by VAN DER GIESSEN [1989c] on the plastic spin during a simple shear deformation process. This model may be regarded as an idealization of the polycrystal models discussed in the foregoing and was used to illustrate and supplement the PIOS theory (VAN DER GIESSEN [1989c,1990]). In the present scope, the presentation is necessarily very brief; further details will be given elsewhere.

The model is based on the rigid-plastic layered material, discussed in Sec. V.1, which deforms mainly by slip along its parallel layers (see Fig. 2). It is recalled that when such material is subjected to simple shear, the orientation α of the layers in the course of the process is given by eqn (31).

Let us consider an overlay or fraction type of model in which each individual component or fraction is constituted by such a layered medium. All components will be differently oriented, but they are assumed to be subjected to the same overall simple shear deformation. This is analogous to the Taylor assumption in polycrystal models. In effect, we will consider the limiting model which consists of an infinite number of components. Hence there exists at any instant a continuous distribution of orientations. A so-called orientation distribution function (ODF) denoted by $n(\alpha,\gamma)$ is introduced such that the relative density of components which, at a shear γ, have an orientation between α and $\alpha + d\alpha$ is given by $n(\alpha,\gamma)d\alpha$. By definition we have $\int_0^\pi n(\alpha,\gamma)\, d\alpha = 1$, so that a uniform distribution corresponding to isotropic properties is characterized by $n(\alpha,\gamma) = 1/\pi$.

Simple shear of the assemblage thus obtained occurs by multiple slip, where the shear rate and the accompanying rate of change of the orientation of the individual components will depend on their current orientation. During a shear increment $d\gamma$, components pass from one orientation interval to the next. The resulting change of the ODF is then given through the continuity equation

$$n' + \frac{\partial}{\partial \alpha}(n\alpha') = 0, \tag{35}$$

where $(\,)' = \partial/\partial\gamma$ and where α' is determined by virtue of eqn (31). The solution of this differential equation for the given initial value $n(\alpha,0)$ reads

$$\frac{n(\alpha,\gamma)}{n(\alpha,0)} = [\sin^2\alpha + (\cos\alpha - \gamma\sin\alpha)^2]^{-1}. \tag{36}$$

In Fig. 3 the ODF is shown for two values of γ and for an initially uniform orientation distribution. It is clearly seen how, by the process of multiple slip, the individual components gradually rotate toward the horizontal axis. This mimics the development of texture in real polycrystalline materials. Despite the high degree of simplicity of the model, the ODF shows a remarkable similarity with that obtained by HARREN and ASARO [1989] for simple shear with a full detailed finite element analysis of a polycrystalline aggregate.

It is now a straightforward matter to evaluate the overall plastic spin $\bar{\omega}^p$ as the weighted average

$$\bar{\omega}^p = \int_0^\pi \omega^p(\alpha) n(\alpha,\gamma)\, d\alpha \tag{37}$$

of the plastic spin $\omega^p(\alpha)$ for components with current orientation α. This is equivalent to MANDEL's [1982] result, eqn (34), for the present situation. By noting that the shear rate on planes inclined at α is given by $\dot{\gamma}\cos 2\alpha$, the plastic spin in the $x^1 - x^2$ plane is found from the single slip expression (24b) as $\omega^p_{12}(\alpha) = \frac{1}{2}\dot{\gamma}\cos 2\alpha$. It turns out now

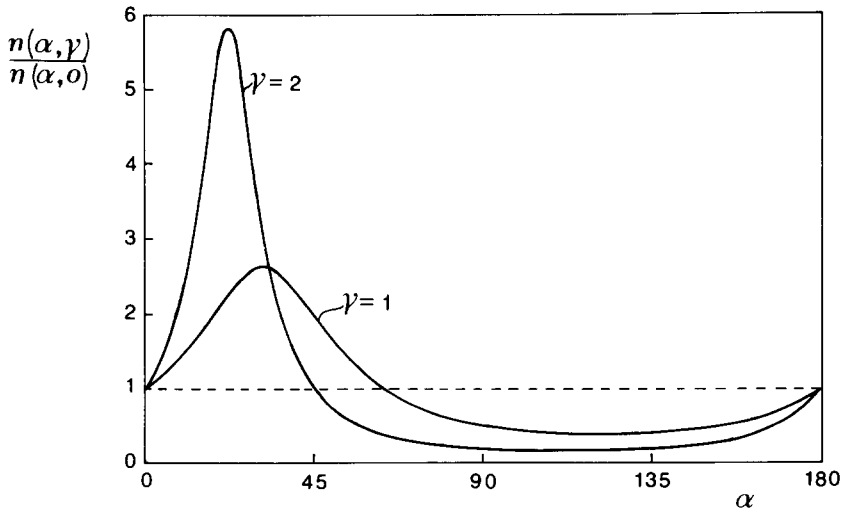

Fig. 3. Orientation distribution functions $n(\alpha,\gamma)$ for various shears. The initial distribution was taken to be uniform, that is, $n(\alpha,0) = 1/\pi$.

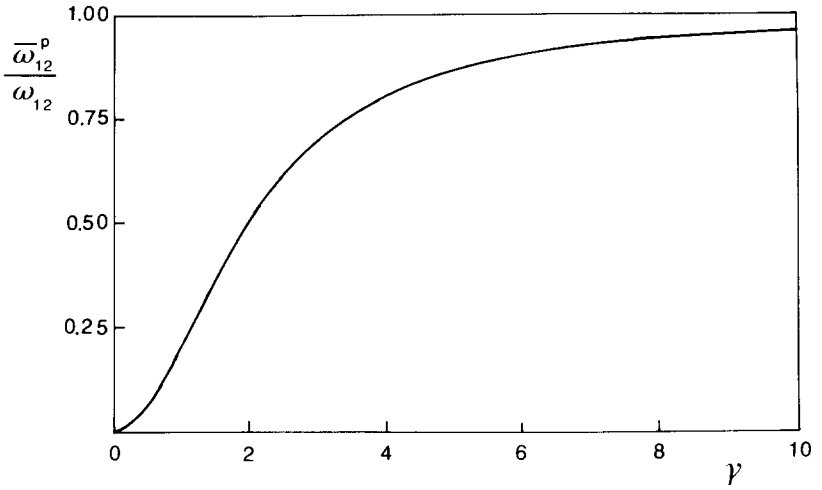

Fig. 4. Average plastic spin versus shear strain according to multiple slip model. The initial distribution was taken to be uniform.

that it is possible to obtain a closed-form solution for $\bar{\omega}_{12}^{p}$ according to (37); it takes the surprisingly simple form

$$\bar{\omega}_{12}^{p}/\omega_{12} = \gamma^2/(\gamma^2 + 4) \tag{38}$$

since $\omega_{12} = \frac{1}{2}\dot{\gamma}$. This result is plotted in Fig. 4.

Comparing this with the plastic spin results in Fig. 1, it is seen that there is a broad correlation with the kinematic hardening predictions of DAFALIAS [1985a] for $\rho = 1$, PAULUN and PECHERSKI [1987] as well as VAN DER GIESSEN [1989b]. It is noted that VAN DER GIESSEN [1989c] found that the multiple slip model predicted the development of strongly nonsymmetric micro-stresses. Although the overall material behavior of the multiple slip model will not exhibit pure kinematic hardening characteristics, the present results support the plastic spin predictions of these phenomenological models. On the other hand, recalling that these kinematic hardening models were derived from rather different principles, the present micromechanical results are not able to show a preference of either one of these principles. Further micromechanical studies of the plastic spin, also using detailed polycrystal models, are recommended.

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REFERENCES

1953 KOITER, W.T., "Stress-Strain Relations, Uniqueness and Variational Theorems for Elastic-Plastic Materials with a Singular Yield Surface," Quart. Appl. Math., **11**, 350.

1968 BESSELING, J.F., "A Thermodynamic Approach to Rheology," in PARKUS, H. and SEDOV, L.I. (eds.), Proc. of IUTAM Symp. on Irreversible Aspects of Continuum Mechanics, Springer-Verlag, Wien, 16-53.

1969 LEE, E.H., "Elastic-Plastic Deformation at Finite Strains," J. Appl. Mech., **36**, 1.

1970 WANG, C.C., "A New Representation Theorem for Isotropic Functions: An Answer to Professor G.F. Smith's Criticism of My Paper on Representations for Isotropic Functions," Arch. Rat. Mech. Anal., **36**, 198.
1971 MANDEL, J., "Plasticité Classique et Viscoplasticité," CISM Lecture Notes No. 97, Udine, Springer-Verlag, Wien.
1971 KRATOCHVIL, J., "Finite-Strain Theory of Crystalline Elastic-Inelastic Materials," J. Appl. Phys., **42**, 1104.
1971 RICE, J.R., "Inelastic Constitutive Relations for Solids: an Internal Variable Theory and its Applications to Metal Plasticity," J. Mech. Phys. Solids, **19**, 433.
1972 HILL, R., "On Constitutive Macro-Variables for Heterogeneous Solids at Finite Strain," Proc. Roy. Soc. London, **A326**, 131.
1973 MANDEL, J., "Equations Constitutives et Directeurs dans les Milieux Plastiques et Viscoplastiques," Int. J. Solids Structures, **9**, 725.
1974 FARDSHISHEH, F. and ONAT, E.T., "Representation of Elastoplastic Behavior by Means of State Variables," in SAWCZUK, A. (ed.), Problems of Plasticity, Noordhoff, 89–115.
1974 HAHN, H.T., "A Finite-Deformation Theory of Plasticity," Int. J. Solids Structures, **10**, 111.
1975 HALPHEN, B., "Sur le Champ des Vitesses en Thermoplasticité Finie," Int. J. Solids Structures, **11**, 947.
1977 BIOT, M.A., "Variational Lagrangian-Thermodynamics of Nonisothermal Strain Mechanics of Porous Solids and Thermomolecular Diffusion," Int. J. Solids Structures, **13**, 579.
1977 ZIEGLER, H., An Introduction to Thermomechanics, North-Holland, Amsterdam.
1979 BERVEILLER, M. and ZAOUI, A., "An Extension of the Self Consistent Scheme to Plastically Flowing Polycrystals," J. Mech. Phys. Solids, **26**, 325.
1982 HAVNER, K.S., "The Theory of Finite Plastic Deformation of Crystalline Solids," in HOPKINS, H.G. and SEWELL, M.J. (eds.), Mechanics of Solids, Pergamon Press, 265–302.
1982 MANDEL, J., "Définition d'un Repère Privilégié Pour l'Etude des Transformations Anélastiques du Polycrystal," J. Méc. Théo. Appl., **1**, 7.
1982 NAGTEGAAL, J.C. and DE JONG, J.E., "Some Aspects of Nonisotropic Workhardening in Finite Strain Plasticity," in LEE, E.H. and MALLETT R.L. (eds.), Plasticity of Metals at Finite Strain: Theory, Computation and Experiment, Stanford Univ./R.P.I., 65–102.
1982 NEMAT-NASSER, S., "On Finite Deformation Elasto-Plasticity," Int. J. Solids Structures, **18**, 857.
1983 ASARO, R.J., "Micromechanics of Crystals and Polycrystals," Adv. Appl. Mech., **23**, 1.
1983 DAFALIAS, Y.F., "Corotational Rates for Kinematic Hardening at Large Plastic Deformations," J. Appl. Mech., **50**, 561.
1983 LORET, B., "On the Effects of Plastic Rotation in the Finite Deformation of Anisotropic Elastoplastic Materials," Mechanics of Materials, **2**, 287.
1984 BIOT, M.A., "New Variational-Lagrangian Irreversible Thermodynamics with Application to Viscous Flow, Reaction-Diffusion, and Solid Mechanics," Adv. Appl. Mech., **24**, 1.
1984 CANOVA, G.R., KOCKS, U.F. and JONAS, J.J., "Theory of Torsion Texture Development," Acta Metall., **32**, 211.
1984 DAFALIAS, Y.F., "The Plastic Spin Concept and a Simple Illustration of Its Role in Finite Plastic Transformations," Mechanics of Materials, **3**, 223.
1984 ONAT, E.T., "Shear Flow of Kinematically Hardening Rigid-Plastic Materials," in DVORAK, G.J. and SHIELD, R.T. (eds.), Elsevier, 311–323.
1985 ASARO, R.J. and NEEDLEMAN, A., "Texture Development and Strain Hardening in Rate Dependent Polycrystals," Acta Metall., **33**, 923.
1985 BESSELING, J.F., "Models of Metal Plasticity: Theory and Experiment," in SAWCZUK, A. and BIANCHI, G. (eds.), Plasticity Today: Modelling, Methods and Applications, Elsevier Appl. Sci. Publ., London/New York, 97–113.
1985a DAFALIAS, Y.F., "A Missing Link in the Macroscopic Constitutive Formulation of Large Plastic Deformations," in SAWCZUK, A. and BIANCHI, G. (eds.), Plasticity Today: Modelling, Methods and Applications, Elsevier Appl. Sci. Publ., London/New York, 135–151.
1985b DAFALIAS, Y.F., "The Plastic Spin," J. Appl. Mech., **52**, 865.
1987 AGAH-TEHRANI, A., LEE, E.H., MALLETT, R.L. and ONAT, E.T., "The Theory of Elastic-Plastic Deformation at Finite Strain with Induced Anisotropy Modeled as Combined Isotropic-kinematic Hardening," J. Mech. Phys. Solids, **35**, 519.
1987 AIFANTIS, E.C., "The Physics of Plastic Deformation," Int. J. Plasticity, **3**, 211.
1987 BAMMANN, D.J. and AIFANTIS, E.C., "A Model for Finite-Deformation Plasticity," Acta Mechanica, **69**, 97.
1987 DAFALIAS, Y.F., "Issues on the Constitutive Formulation at Large Elastoplastic Deformations, Part 1: Kinematics," Acta Mechanica, **69**, 119.
1987 PAULUN, J.E. and PECHERSKI, R.B., "Remarks on the Description of Anisotropic Hardening in Finite Deformation Plasticity," Archives of Mechanics (to appear) [The 4th Bilateral Symposium PRL-BRD on Mechanics of Inelastic Solids and Structures, 13–19 September, Kraków-Mogilany].
1987 ZIEGLER, H. and WEHRLI, C., "On a Principle of Maximal Rate of Entropy Production," J. Non-Equilib. Thermodyn., **12**, 229.

1988 DAFALIAS, Y.F., "Issues on the Constitutive Formulation at Large Elastoplastic Deformations, Part 2: Kinetics," Acta Mechanica, **73**, 121.
1988 PECHERSKI, R.B., "The Plastic Spin Concept and the Theory of Finite Plastic Deformations with Induced Anisotropy," Arch. Mech., **40**, 807.
1988 PERZYNA, P., "Temperature and Rate Dependent Theory of Plasticity of Crystalline Solids," Revue Phys. Appl., **23**, 445.
1988 TOKUDA, M. and YAMADA, K., "Inelastic Constitutive Equations of Polycrystalline Metals Subjected to Finite Deformation, Part I: Effect of Grain Rotation," Int. J. Plasticity, **4**, 47.
1989 HARREN, S.V. and ASARO, R.J., "Nonuniform Deformations in Polycrystals and Aspects of the Validity of the Taylor Model," J. Mech. Phys. Solids, **37**, 191.
1989 HARREN, S.V., LOWE, T., ASARO, R.J. and NEEDLEMAN, A., "Analysis of Large-Strain Shear in Rate-Dependent FCC Polycrystals: Correlation of Micro and Macromechanics," Phil. Trans. Roy. Soc. London, **A328**, 443.
1989a VAN DER GIESSEN, E., "Continuum Models of Large Deformation Plasticity—Part I: Large Deformation Plasticity and the Concept of a Natural Reference State," Eur. J. Mech., A/Solids, **8**, 15.
1989b VAN DER GIESSEN, E., "Continuum Models of Large Deformation Plasticity—Part II: A Kinematic Hardening Model and the Concept of a Plastically Induced Orientational Structure," Eur. J. Mech., A/Solids, **8**, 89.
1989c VAN DER GIESSEN, E., "On a Continuum Representation of Deformation-Induced Anisotropy," in FAN, J. and MURAKAMI, S. (eds.), Advances in Constitutive Laws for Engineering Materials, Pergamon Press, Oxford, 503–509.
1990 DAFALIAS, Y.F. and AIFANTIS, E.C., "On the microscopic origin of the plastic spin," Acta Mechanica, **82**, 31.
1990 VAN DER GIESSEN, E., "A Model of Anisotropically Hardening Materials Based upon the Concept of a Plastically-Induced Orientational Structure," in BOEHLER, J.P. (ed.), Yielding, Damage, and Failure of Anisotropic Solids, EGF5, Mech. Engrg. Publ., London, 187–198.

Faculty of Mechanical Engg. and Marine Tech.
Delft University of Technology
P.O. Box 5033
2600 GA Delft, The Netherlands

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